On the convex hull of symmetric stable processes

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Abstract

Let $\alpha \in (1, 2]$ and X be an $\mathbb{R}^d$-valued α-stable process with independent and symmetric components starting in 0. We consider the closure S_t of the path described by X on the interval $[0, t]$ and its convex hull Z_t. The first result of this paper provides a formula for certain mean mixed volumes of Z_t and in particular for the expected first intrinsic volume of Z_t. The second result deals with the asymptotics of the expected volume of the stable sausage $Z_t + B$ (where B is an arbitrary convex body with interior points) as $t \to 0$.

Keywords: stable process, convex hull, mixed volume, intrinsic volume, stable sausage, Wiener sausage, mean body

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1 Introduction and main results

For fixed $\alpha \in (1, 2]$ and fixed integer $d \geq 1$ we consider an $\mathbb{R}^d$-valued stochastic process $X \equiv (X(t))_{t \geq 0} = (X_1(t), \ldots, X_d(t))_{t \geq 0}$, defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, such that the components $X_j := (X_j(t))_{t \geq 0}$, $j \in \{1, \ldots, d\}$, are independent α-stable symmetric Lévy processes with scale parameter 1 starting in 0. The characteristic function of $X_j(t)$ is given by

$$\mathbb{E} \exp[isX_j(t)] = \exp[-t|s|^\alpha], \quad s \in \mathbb{R}, t \geq 0, \tag{1.1}$$

cf. [8, Section 1.3]. This implies that X is *self-similar* in the sense that $(X(st))_{s \geq 0} \overset{d}{=} t^{1/\alpha} X$ for any $t > 0$, see [8, Example 7.1.3] and [4, Chapter 15]. We assume that X is right-continuous with left-hand limits (rcll). For $t \geq 0$, let S_t be the closure of the path $S_t^0 := \{X(s) : 0 \leq s \leq t\}$ and let Z_t denote the convex hull of S_t. These are random closed sets. We abbreviate $Z := Z_1$. By self-similarity

$$Z_t \overset{d}{=} t^{1/\alpha} Z, \quad t > 0. \tag{1.2}$$

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If $\alpha = 2$ then X is a standard Brownian motion. A classical result of [12] for planar Brownian motion says that

$$\mathbb{E}V_1(Z) = \sqrt{2\pi}, \tag{1.3}$$

where $V_1(K)$ denotes half the circumference of a convex set $K \subset \mathbb{R}^2$. Our first aim in this paper is to formulate and to prove such a result for arbitrary $\alpha \in (1, 2]$ and arbitrary dimension d. In fact we also consider more general geometric functionals. A *convex body* (in $\mathbb{R}^d$) is a non-empty compact and convex subset of $\mathbb{R}^d$. We let $V(K_1, \dots, K_d)$, denote the mixed volumes of convex bodies $K_1, \dots, K_d \subset \mathbb{R}^d$ [9, Section 5.1]. These functionals are symmetric in $K_1, \dots, K_d$ and we have for any convex bodies $K, B \subset \mathbb{R}^d$

$$V_d(K + tB) = \sum_{j=0}^{d} \binom{d}{j} t^{d-j} V(K[j], B[d-j]), \quad t \geq 0, \tag{1.4}$$

where V_d is Lebesgue measure, $tB := \{tx : x \in B\}$, $B + C := \{x + y : x \in B, y \in C\}$ is the *Minkowski sum* of two sets $B, C \subset \mathbb{R}^d$, and $V(K[j], B[d-j])$ is the mixed volume of $K_1, \dots, K_d$ in case $K_1 = \ldots = K_j = K$ and $K_{j+1} = \ldots = K_d = B$. The jth *intrinsic volume* $V_j(K)$ of a convex body K is given by

$$V_j(K) = \frac{\binom{d}{j}}{\kappa_{d-j}} V(K[j], B^d[d-j]), \quad j = 0, \dots, d, \tag{1.5}$$

where B^d is the Euclidean unit ball in $\mathbb{R}^d$, and κ_j is the j-dimensional volume of B^j. In particular, $V_d(K)$ is the volume of K, $V_{d-1}(K)$ is half the surface area, $V_{d-2}(K)$ is proportional to the integral mean curvature, $V_1(K)$ is proportional to the mean width of K, and $V_0(K) = 1$. (If $d = 2$ then $V_1(K)$ has been discussed at (1.3).) A geometric interpretation of $V(K_1, \dots, K_d)$ in the case $K_1 = \ldots = K_{d-1} = B$ is provided by (1.4):

$$V(B, \dots, B, K) = \lim_{r \to 0} r^{-1}(V_d(B + rK) - V_d(B)). \tag{1.6}$$

For any $p \geq 1$ define $B_p := \{u \in \mathbb{R}^d : \|u\|_p \leq 1\}$ as the unit ball with respect to the L_p-norm $\|(u_1, \dots, u_d)\|_p := (|u_1|^p + \ldots + |u_d|^p)^{1/p}$. Finally we introduce the constant

$$c_\alpha := \frac{\alpha}{2} \mathbb{E}|X_1(1)|, \tag{1.7}$$

Since $\alpha > 1$, this constant is finite, see [8, Property 1.2.16]. A direct calculation shows that

$$c_2 = \sqrt{\frac{2}{\pi}}. \tag{1.8}$$

In the case of $\alpha < 2$ we are not aware of an explicit expression for c_α.

Theorem 1.1. *Let $K_1, \dots, K_{d-1} \subset \mathbb{R}^d$ be convex bodies. Then*

$$\mathbb{E}V(K_1, \dots, K_{d-1}, Z) = c_\alpha V(K_1, \dots, K_{d-1}, B_{\alpha'}), \tag{1.9}$$

where $1/\alpha + 1/\alpha' = 1$.

Remark 1.2. By the scaling relation (1.2) and the homogeneity property of mixed volumes [9, (5.1.24)] the identity (1.9) can be generalized to

$$\mathbb{E}V(K_1, \dots, K_{d-1}, Z_t) = c_\alpha t^{1/\alpha} V(K_1, \dots, K_{d-1}, B_{\alpha'}). \tag{1.10}$$

A similar remark applies to all results of this paper.

The proof of Theorem 1.1 relies on the fact that

$$\mathbb{E}h(Z, u) = h(B_{\alpha'}, u), \quad u \in S^{d-1}, \tag{1.11}$$

where S^{d-1} denotes the unit sphere,

$$h(K, u) := \max\{\langle x, u\rangle : x \in K\}, \quad u \in S^{d-1},$$

is the *support function* of a convex body K, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^d$. This means that $B_{\alpha'}$ is the *mean body* of Z [10, p. 146], or the *selection expectation* of Z [7, Theorem 2.1.22].

The next corollary provides a direct generalization of (1.3).

Corollary 1.3. *Assume that X is a standard-Brownian motion in $\mathbb{R}^d$. Then*

$$\mathbb{E}V_1(Z) = \frac{d\sqrt{2}\Gamma(\frac{d-1}{2} + 1)}{\Gamma(\frac{d}{2} + 1)} \tag{1.12}$$

In the case of Brownian motion it is possible to calculate the expectation of the second intrinsic volume $V_2(Z)$ of Z.

Proposition 1.4. *Assume that X is a standard-Brownian motion in $\mathbb{R}^d$. Then*

$$\mathbb{E}V_2(Z) = (d-1)\frac{\pi}{2}.$$

Our second theorem deals with the asymptotic behaviour of the expected volume of the *stable sausage* $S_t + B$ as $t \to 0$, where B is a convex body. Our result complements classical results on the asymptotic behaviour of $\mathbb{E}V_d(S_t + B)$ as $t \to \infty$, cf. [11] for the case of Brownian motion and [3] for the case of more general stable processes.

Theorem 1.5. *Let B be a convex body with non-empty interior and let α' be as in Theorem 1.1. Then*

$$\lim_{t\to 0} t^{-1/\alpha}(\mathbb{E}V_d(S_t + B) - V_d(B)) = dc_\alpha V(B, \dots, B, B_{\alpha'}).$$

In the case $\alpha = 2$ the random set $S_t + B$ is known as *Wiener sausage*. Even then Theorem 1.5 seems to be new:

Corollary 1.6. *Assume that X is a Brownian motion and let B be a convex body with non-empty interior. Then*

$$\lim_{t\to 0} t^{-1/2}(\mathbb{E}V_d(S_t + B) - V_d(B)) = \frac{d\sqrt{2}}{\sqrt{\pi}} V(B, \dots, B, B^d).$$

In the case $B = B^d$ the limit equals $2\sqrt{2}\pi^{(d-1)/2}/\Gamma(d/2)$.

Remark 1.7. In the special case $d = 3$ and $\alpha = 2$ we have (see [11])

$$\mathbb{E}\, V_3(S_t + rB^3) = \frac{4}{3}\pi r^3 + 4\sqrt{2\pi} r^2 \sqrt{t} + 2\pi r t \tag{1.13}$$

for any $r, t \geq 0$. The term constant in t clearly allows a geometric interpretation as $V_3(rB^3)$. Now we are able to give a geometric interpretation of the coefficient of $\sqrt{t}$ as well.

From (1.13) we get

$$\lim_{t\to 0} t^{-1/2}(\mathbb{E}V_3(S_t + rB^3) - V_3(rB^3)) = \lim_{t\to 0} t^{-1/2}(4\sqrt{2\pi} r^2\sqrt{t} + 2\pi r t) = 4\sqrt{2\pi} r^2.$$

On the other hand from the proof of Theorem 1.5 one can see

$$\lim_{t\to 0} t^{-1/2}(\mathbb{E}V_3(S_t + rB^3) - V_3(rB^3)) = 3\mathbb{E}V(rB^3, rB^3, Z).$$

By (1.5) and the homogenity property of mixed volumes (see e.g. [9, (5.1.24)]) we have

$$3V(rB^3, rB^3, Z) = r^2\kappa_2 V_1(Z).$$

Altogether this is

$$4\sqrt{2\pi} r^2 = r^2 \kappa_2 \mathbb{E}V_1(Z).$$

2 Proofs

We need the following measurability property of the closure S_t of $\{X(s) : 0 \leq s \leq t\}$ and its convex hull Z_t, refering to [7, 10] for the notion of a *random closed set.*

Lemma 2.1. *For any $t \geq 0$, S_t and Z_t are random closed sets.*

Proof: To prove the first assertion it is enough to show that $\{S_t \cap G = \emptyset\}$ is measurable for any open $G \subset \mathbb{R}^d$, see [10, Lemma 2.1.1]. But since X is rccl it is clear that $S_t \cap G = \emptyset$ iff $X(u) \notin G$ for all rational numbers $u \leq t$. The second assertion is implied by [10, Theorems 12.3.5,12.3.2]. □

The previous lemma implies, for instance, that $V(K_1, \ldots, K_{d-1}, Z_t)$ and $V_d(S_t + B)$ are random variables, see e.g. [9, p. 275] and [10, Theorem 12.3.5 and Theorem 12.3.6].

Proof of Theorem 1.1: By [9, (5.1.18)] we have that

$$V(K_1, \ldots, K_{d-1}, K) = \frac{1}{d}\int_{S^{d-1}} h(K, u)\, S(K_1, \ldots, K_{d-1}, du) \tag{2.1}$$

holds for every convex body $K \subset \mathbb{R}^d$, where $S(K_1, \ldots, K_{d-1}, \cdot)$ is the *mixed area measure* of $K_1, \ldots, K_{d-1}$, see [9, Section 4.2]. From (2.1) and Fubini's theorem we obtain that

$$\mathbb{E}V(K_1, \ldots, K_{d-1}, Z) = \frac{1}{d}\int_{S^{d-1}} \mathbb{E}h(Z, u)\, S(K_1, \ldots, K_{d-1}, du). \tag{2.2}$$

For any $u \in S^{d-1}$ we have

$$\begin{aligned}\mathbb{E}h(Z,u) &= \mathbb{E}\max\{\langle x,u\rangle : x \in Z_1\}\\ &= \mathbb{E}\sup\{\langle x,u\rangle : x \in S_1^0\}\\ &= \mathbb{E}\sup\{\langle X(s),u\rangle : s \in [0,1]\}.\end{aligned}$$

It follows directly from (1.1) that the process $\langle X,u\rangle$ has the same distribution as $\|u\|_\alpha X_1$. By [1, Theorem 4a], $\sup\{X_1(s) : s \in [0,1]\}$ has a finite expectation. Differentiating equation (7b) in [1] (Spitzer's identity in continuous time), one can easily show that

$$\mathbb{E}\sup\{X_1(s) : s \in [0,1]\} = \alpha \mathbb{E}X_1(1)^+,$$

where $a^+ := \max\{0,a\}$ denotes the positive part of a real number a. Since $X_1(1)$ has a symmetric distribution and $\mathbb{P}(X_1(1) = 0) = 0$ (stable distributions have a density) we have $\mathbb{E}|X_1(1)| = 2\mathbb{E}X_1(1)^+$ and it develops that $\mathbb{E}h(Z,u) = c_\alpha \|u\|_\alpha$, with c_α given by (1.7). Inserting this result into (2.2) gives

$$\mathbb{E}V(K_1,\dots,K_{d-1},Z) = \frac{c_\alpha}{d}\int_{S^{d-1}} \|u\|_\alpha \, S(K_1,\dots,K_{d-1},du). \tag{2.3}$$

By [9, Remark 1.7.8], $\|u\|_\alpha$ is the support function of the *polar body*

$$B_\alpha^* := \{x \in \mathbb{R}^d : \langle x,u\rangle \le 1 \text{ for all } u \in B_\alpha\}$$

of B_α. Using the Hölder inequality, it is straightforward to check that $B_\alpha^* = B_{\alpha'}$, where $1/\alpha + 1/\alpha' = 1$. Using this fact as well as (2.1) in (2.3), we obtain the assertion (1.9). □

Proof of Corollary 1.3: Since $\alpha = 2$ we have $\alpha' = 2$ and $B_{\alpha'} = B^d$. By Theorem 1.1 and (1.5),

$$\mathbb{E}V_1(Z) = \mathbb{E}\frac{d}{\kappa_{d-1}}V(B^d,\dots,B^d,Z) = \frac{c_2 d}{\kappa_{d-1}}V(B^d,\dots,B^d,B^d) = \frac{c_2 d\kappa_d}{\kappa_{d-1}}, \tag{2.4}$$

where we have used that $V(B^d,\dots,B^d) = V_d(B^d)$. Using (1.8) and the well-known formula $\kappa_d = \pi^{d/2}/\Gamma(d/2+1)$ in (2.4), we obtain the result. □

Proof of Proposition 1.4: By Kubota's formula (see e.g. [9, (5.3.27)]) we have

$$V_2(Z) = \frac{d(d-1)\kappa_d}{2\kappa_2\kappa_{d-2}}\int_{G_2} V_2(Z|L)\,\nu_2(dL),$$

where G_2 denotes the set of all 2-dimensional linear subspaces of $\mathbb{R}^d$, ν_2 is the Haar measure on G_2 with $\nu_2(G_2) = 1$ and $Z|L$ denotes the image of Z under the orthogonal projection onto the linear subspace L. By Fubini's theorem,

$$\mathbb{E}V_2(Z) = \frac{d(d-1)\kappa_d}{2\kappa_2\kappa_{d-2}}\int_{G_2} \mathbb{E}V_2(Z|L)\,\nu_2(dL).$$

The spherical symmetry of Brownian motion implies that $\mathbb{E}V_2(Z|L)$ does not depend on L. Assume that $L = \{(x_1,x_2,0,\dots,0) : x_1,x_2 \in \mathbb{R}\}$. Now it is clear from the definition of

the d-dimensional Brownian motion that the random closed set $Z|L$ is the convex hull of a Brownian path in L. By Remark (a) in [2, p. 149] (see also [6]) we have $\mathbb{E}V_2(Z|L) = \pi/2$. Therefore,

$$\mathbb{E}V_2(Z) = \frac{d(d-1)\kappa_d}{2\kappa_2\kappa_{d-2}}\frac{\pi}{2}$$

and the result follows by a straightforward calculation. $\square$

Proof of Theorem 1.5: By self-similarity and the dominated convergence theorem, on whose conditions we will comment below, we have

$$\begin{aligned}\lim_{t\to 0} t^{-1/\alpha}(\mathbb{E}V_d(S_t+B) - V_d(B)) &= \lim_{t\to 0} t^{-1/\alpha}(\mathbb{E}V_d(t^{1/\alpha}S_1+B) - V_d(B)) \\ &= \lim_{t\to 0} t^{-1}(\mathbb{E}V_d(tS_1+B) - V_d(B)) \\ &= \mathbb{E}\lim_{t\to 0} t^{-1}(V_d(tS_1+B) - V_d(B)). \end{aligned} \tag{2.5}$$

In order to justify the application of the dominated convergence theorem, put

$$Y_j = \sup\{X_j(s) : s \in [0,1]\}, \qquad \tilde{Y}_j = \inf\{X_j(s) : s \in [0,1]\}, \quad j = 1, \dots, d.$$

As noted in the proof of Theorem 1.1, Y_j has a finite expectation. Since $-\tilde{Y}_j$ has the same distribution as Y_j, $\tilde{Y}_j$ has a finite expectation as well. From (1.4) we obtain for all $t \in (0,1]$ that

$$\begin{aligned} t^{-1}V_d(tS+B) - V_d(B) &\le t^{-1}(V_d(tZ+B) - V_d(B)) \\ &= \sum_{j=0}^{d-1}\binom{d}{j} t^{d-j-1} V(B[j], Z[d-j]) \\ &\le \sum_{j=0}^{d}\binom{d}{j} V(B[j], Z[d-j]) \\ &= V_d(Z+B). \end{aligned}$$

Furthermore,

$$Z + B \subset \mathop{\times}_{j=1}^{d} [\tilde{Y}_j - h_B(-e_j), Y_j + h_B(e_j)],$$

where e_j denotes the jth unit vector. It follows that

$$t^{-1}(V_d(tS_1+B) - V_d(B)) \le \prod_{j=1}^{d}\left(Y_j + h_B(e_j) - \tilde{Y}_j + h_B(-e_j)\right), \quad t \in (0,1].$$

This is a product of independent random variables with finite expected values and hence has finite expected value.

By [5, Corollary 3.2 (2)] we have

$$\lim_{t\to 0} t^{-1}(V_d(tS_1+B) - V_d(B)) = \int_{S^{d-1}} h(Z,u)\, S(B, \dots, B, du),$$

and using Theorem 1.1 we conclude from (2.5)

$$\begin{aligned}\lim_{t\to 0} t^{-1/\alpha}\mathbb{E}(V_d(S_t+B)-V_d(B)) &= \mathbb{E}\int_{S^{d-1}} h(Z,u)\, S(B,\dots,B,du)\\ &= d\mathbb{E}V(B,\dots,B,Z)\\ &= dc_\alpha V(B,\dots,B,B_{\alpha'}). \qquad \square\end{aligned}$$

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